

Wave equations on non-smooth space-times

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Abstract

We consider wave equations on Lorentzian manifolds in case of low regularity. We first extend the classical solution theory to prove global unique solvability of the Cauchy problem for distributional data and right hand side on smooth globally hyperbolic space-times. Then we turn to the case where the metric is non-smooth and present a local as well as a global existence and uniqueness result for a large class of Lorentzian manifolds with a weakly singular, locally bounded metric in Colombeau's algebra of generalized functions.

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1 Introduction

In this note we are concerned with wave equations on Lorentzian manifolds, with a particular interest in the Cauchy problem in non-smooth situations. To achieve a self-contained presentation we first give a brief account on the classical solution theory in the smooth case, in particular discussing local well-posedness as well as global well-posedness on globally hyperbolic manifolds. We first extend this theory to the case of distributional data and right hand sides. Then we turn to the case where the metric is non-smooth, that is we deal with normally hyperbolic operators with coefficients of low regularity. Actually the regularity class of the metric which we have in mind is below $\mathcal{C}^{1,1}$ (i.e., the first derivative locally Lipschitz)—the largest class where standard differential geometric results such as existence and uniqueness of geodesics remain valid, and also below the Geroch-Traschen class of metrics—the largest class that allows for a consistent distributional treatment ([16, 26]). It is evident that no consistent distributional solution concept is available for these equations. Therefore we consider a large

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class of weakly singular, locally bounded metrics in the setting of generalized global analysis and nonlinear distributional geometry ([19, Sec. 3.2]) based on Colombeau algebras of generalized functions ([10, 11]). In particular, we present a local existence and uniqueness result in the spirit of [17] and extend it to a global result on space-times with a generalized metric that allows for a suitable globally hyperbolic metric splitting.

This line of research has drawn some motivation from general relativity: In [8] Chris Clarke suggested to replace the standard geometric definition of singularities by viewing them as obstructions to the well-posedness of the initial value problem for a scalar field. However, for many relevant examples as e.g. impulsive gravitational waves, cosmic strings, and shell crossing singularities which have a metric of low regularity the Cauchy problem cannot consistently be formulated in distribution theory and one has to use a more sophisticated solution concept. In case of shell crossing singularities Clarke himself used a cleverly designed weak solution concept to argue for local solvability of the wave equation ([9]). On the other hand Vickers and Wilson ([33]) used Colombeau generalized functions to show local well-posedness of the wave equation in conical space-times modelling a cosmic string. This result has been generalized to a class of locally bounded space-times in [17], also see [27] for the static case and [20] for an extension to non-scalar equations.

This work is organized in the following way. In Section 2 we recall the classical theory of normally hyperbolic operators on smooth Lorentzian manifolds, thereby essentially following a recent book by Bär et. al. ([1]). We extend their global existence and uniqueness result (Th. 2.7) to the case of distributional data and right hand side in Section 3. In Section 4 we recall the necessary background from global analysis based on Colombeau generalized functions ([19]). We devote Section 5 to presenting a variant of the local existence and uniqueness theorem for the wave equation in weakly singular space-times of [17]. Finally, in Section 6 we extend this result to a global theorem for weakly singular space-times which allow for a suitable globally hyperbolic splitting of the metric.

2 A review of wave equations on smooth Lorentzian manifolds

In this section we present the solution theory for wave equations on smooth Lorentzian manifolds with smooth right hand side and data (cf. (3) below), mainly based on [1, 34] (see also [14]). We will at once formulate the theory for normally hyperbolic operators on sections of a vector bundle E over some Lorentzian manifold M .

We first fix some notations. Throughout we will assume (M, g) to be a smooth, connected, time-oriented Lorentz manifold and $E \rightarrow M$ a smooth vector bundle over M . By $\Gamma(M, E)$ or $\Gamma(E)$ we denote the smooth sections of $E \rightarrow M$, $\mathcal{D}(M, E)$ is the space of compactly supported sections, and $\Gamma^k(E)$ denotes the spaces of sections of finite differentiability. Given a finite-dimensional vector

space W , the space of W -valued distributions in E , $\mathcal{D}'(M, E, W)$ comprises the continuous linear maps $\mathcal{D}(M, E^*) \rightarrow W$. For example, given $x \in M$, the delta-distribution δ_x is the E_x^* -valued distribution in E given by $\delta_x : \mathcal{D}(M, E^*) \rightarrow E_x^*$, $\varphi \mapsto \varphi(x)$.

A differential operator $P : \Gamma(E) \rightarrow \Gamma(E)$ of second order is called *normally hyperbolic* if its principal symbol satisfies $\sigma_P(\xi)|_x = -\langle \xi, \xi \rangle \cdot \text{id}_{E_x}$ ($\xi \in T_x^*M$). If E is equipped with a linear connection ∇ then an important example of a normally hyperbolic operator is given by the connection d'Alembertian

$$\square^\nabla : \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes E) \xrightarrow{-\text{tr} \otimes \text{id}_E} \Gamma(E),$$

where $\text{tr} : T^*M \otimes T^*M \rightarrow \mathbb{R}$, $\text{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$ is the metric trace. By the Weitzenböck formula, for any normally hyperbolic operator on $\Gamma(E)$ there exists a unique connection ∇ on E and a unique $B \in \Gamma(\text{Hom}(E, E))$ such that $P = \square^\nabla + B$. For any differential operator $D : \Gamma(E) \rightarrow \Gamma(F)$, its formal adjoint $D^* : \Gamma(F^*) \rightarrow \Gamma(E^*)$ is uniquely characterized by

$$\int \psi(D\varphi) dV = \int (D^*\psi)(\varphi) dV$$

for all $\varphi \in \mathcal{D}(M, E)$ and all $\psi \in \mathcal{D}(M, F^*)$. Here dV is the Lorentzian volume density on M .

For notions from causality in Lorentz manifolds our main references are [28, 2]. In particular, for $p, q \in M$, by $p < q$ (resp. $p << q$) we mean that there exists a causal (resp. timelike) future directed curve from p to q . For $A \subseteq \Omega \subseteq M$ we write

$$I_+^\Omega(A) := \{q \in \Omega \mid \exists p \in A \text{ s.t. } q >> p \text{ in } \Omega\}$$

for the relative chronological future of A in Ω , and analogous for $I_-^\Omega(A)$. Also we write

$$J_+^\Omega(A) := \{q \in \Omega \mid \exists p \in A \text{ s.t. } q \geq p \text{ in } \Omega\},$$

for the relative causal future and analogous for $J_-^\Omega(A)$. Here $q \geq p$ means either $p < q$ or $p = q$. A domain Ω is called causal if its closure $\bar{\Omega}$ is contained in a geodesically convex domain Ω' and if for all $p, q \in \bar{\Omega}$ the causal diamond $J_+^M(p) \cap J_-^M(q)$ is a compact subset of Ω' . The manifold M is said to satisfy the *causality condition* (CC) if there are no closed causal curves in M . It satisfies the *strong causality condition* (SCC) if for each $p \in M$ and each neighborhood U of p there exists a neighborhood $V \subseteq U$ such that no causal curve that starts and ends in V can leave U (i.e., there are no 'almost closed' causal curves in M).

A *Cauchy hypersurface* is a subset S of M that is intersected by each inextendible timelike curve exactly once. M is called *globally hyperbolic* if it satisfies SCC and for all $p, q \in M$ the causal diamond $J_+^{\Omega'}(p) \cap J_-^{\Omega'}(q)$ is compact. The following is a very useful characterization of global hyperbolicity, due in its final form to [3].

Theorem 2.1. *For any time-oriented Lorentzian manifold M , the following are equivalent:*

- (i) *M is globally hyperbolic.*
- (ii) *There exists a Cauchy hypersurface in M .*
- (iii) *M is isometric to $(\mathbb{R} \times S, -\beta dt^2 + g_t)$, where $\beta = \beta(t, x)$ is smooth and strictly positive, $t \mapsto g_t$ is a smooth family of Riemannian metrics on S , and each $\{t\} \times S$ is a (smooth) spacelike Cauchy-hypersurface in M .*

We remark that by a recent result ([4]) global hyperbolicity is also characterized by the condition that results from replacing SCC by CC in the above definition.

Turning now to the problem of solving the initial value problem for a normally hyperbolic differential operator on $E \rightarrow M$, we first consider the case where $M = (V, \langle \cdot, \cdot \rangle)$ is a Lorentz vector space of dimension n (later on, the role of V will be played by a tangent space $T_x M$ to M). We denote by $\gamma : V \rightarrow \mathbb{R}$, $\gamma(X) := -\langle X, X \rangle$ the quadratic form associated with $\langle \cdot, \cdot \rangle$. The analytic centerpiece of the entire construction that is to follow is provided by the so-called *Riesz distributions*:

Definition 2.2. *For $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > n$, let*

$$R_{\pm}(\alpha)(X) := \begin{cases} C(\alpha, n) \gamma(X)^{\frac{\alpha-n}{2}} & \text{if } X \in J_{\pm}(0) \\ 0 & \text{else} \end{cases}$$

where $C(\alpha, n) := \frac{2^{1-\alpha} \pi^{\frac{2-n}{2}}}{\Gamma(\frac{\alpha}{2}) \Gamma(\frac{\alpha-n}{2} + 1)}$. $R_{+}(\alpha)$ ($R_{-}(\alpha)$) is called *advanced* (*retarded*) *Riesz distribution on V* .

$R_{\pm}(\alpha)$ is continuous on V , and using the fact that $\square R_{\pm}(\alpha + 2) = R_{\pm}(\alpha)$ for $\operatorname{Re}(\alpha) > n + 2$, $\alpha \mapsto R_{\pm}(\alpha)$ uniquely extends to a holomorphic family of distributions on all of \mathbb{C} . For all α , $\operatorname{supp}(R_{\pm}(\alpha)) \subseteq J_{\pm}(0)$ and $\operatorname{singsupp}(R_{\pm}(\alpha))$ is contained in the boundary $C_{\pm}(0)$ of $J_{\pm}(0)$. Moreover, $R_{\pm}(0) = \delta_0$.

The next step in the construction is to transport the Riesz distributions onto the Lorentz manifold M . To this end, let Ω be a normal neighborhood of $x \in M$ and define the smooth function $\mu_x : \Omega \rightarrow \mathbb{R}$ by $dV = \mu_x \cdot (\exp_x)_*(dz)$, where dz is the standard volume density on $T_x M$, and \exp_x is the exponential map at x . In normal coordinates around x , $\mu_x = \sqrt{|\det g_{ij}|}$. Now we set $R_{\pm}^{\Omega}(\alpha, x) := \mu_x \cdot (\exp_x)_* R_{\pm}(\alpha)$, i.e.,

$$\forall \varphi \in \mathcal{D}(\Omega, \mathbb{C}) : \quad \langle R_{\pm}^{\Omega}(\alpha, x), \varphi \rangle = \langle R_{\pm}(\alpha), (\mu_x \cdot \varphi) \circ \exp_x \rangle.$$

$R_{\pm}^{\Omega}(\alpha, x)$ are called *advanced* (*retarded*) *Riesz distributions on Ω* . The analytical properties of $R_{\pm}(\alpha)$ carry over to the manifold setting, albeit in slightly more involved form, e.g. (setting $\Gamma_x := \gamma \circ \exp_x^{-1}$),

$$\square R_{\pm}^{\Omega}(\alpha + 2, x) = \left(\frac{1}{2\alpha} (\square \Gamma_x - 2n) + 1 \right) R_{\pm}^{\Omega}(\alpha, x) \quad \text{for } \alpha \neq 0. \quad (1)$$

Moreover, $R_{\pm}^{\Omega}(0, x) = \delta_x$, $\text{supp}(R_{\pm}^{\Omega}(\alpha, x)) \subseteq J_{\pm}^{\Omega}(x)$ and $\text{singsupp}(R_{\pm}^{\Omega}(\alpha, x)) \subseteq C_{\pm}^{\Omega}(x)$.

For any normally hyperbolic operator P on $\Gamma(E)$, our aim is to construct a fundamental solution in the following sense:

Definition 2.3. *Let $P : \Gamma(E) \rightarrow \Gamma(E)$ be normally hyperbolic. A distribution $F \in \mathcal{D}'(M, E, E_x^*)$ such that $PF = \delta_x$ (i.e., $\langle F, P^* \varphi \rangle = \varphi(x)$ for all $\varphi \in \mathcal{D}(M, E^*)$) is called a fundamental solution of P at $x \in M$. F is called advanced (retarded) if $\text{supp}(F) \subseteq J_+^M(x)$ ($\subseteq J_-^M(x)$).*

For example, in a Lorentz vector space V as above, $R_{\pm}(2)$ is an advanced (retarded) fundamental solution of \square at 0 since $\square R_{\pm}(2) = R_{\pm}(0) = \delta_0$. In the manifold setting matters are more complicated, as already indicated in (1). We first make the following formal ansatz for a fundamental solution on a normal neighborhood Ω of x :

$$\mathcal{R}_{\pm}(x) := \sum_{k=0}^{\infty} V_x^k R_{\pm}^{\Omega}(2 + 2k, x), \quad (2)$$

where $V_x^k \in \Gamma(\Omega, E \otimes E_x^*)$, the so-called *Hadamard-coefficients*, are to be determined. By formal termwise differentiation one finds that in order for $\mathcal{R}_{\pm}(x)$ to be a fundamental solution, the V_x^k have to satisfy the following transport equations:

$$\nabla_{\text{grad } \Gamma_x} V_x^k - \left(\frac{1}{2} \square \Gamma_x - n + 2k\right) V_x^k = 2k P V_x^{k-1} \quad (k \geq 0)$$

with $V_x^0(x) = \text{id}_{E_x}$. The Hadamard coefficients are therefore uniquely determined as the solution to this problem.

Next, introducing convergence-generating factors into (2) we obtain an approximate fundamental solution $\tilde{\mathcal{R}}(x)$ in the following sense:

$$\exists K_{\pm} \in \Gamma(\bar{\Omega} \times \bar{\Omega}, E^* \boxtimes E) \text{ s.t. } P_{(2)} \tilde{\mathcal{R}}(x) = \delta_x + K_{\pm}(x, \cdot).$$

Here, $E^* \boxtimes E$ denotes the exterior tensor product and $P_{(2)}$ indicates that P acts on the second variable. We now use the K_{\pm} as integral kernels to define for any continuous section u of E^* over $\bar{\Omega}$:

$$(\mathcal{K}_{\pm} u)(x) := \int_{\bar{\Omega}} K_{\pm}(x, y) u(y) dV(y).$$

For Ω a sufficiently small causal domain and any k , $\text{id} + \mathcal{K}_{\pm} : \Gamma^k(\bar{\Omega}, E^*) \rightarrow \Gamma^k(\bar{\Omega}, E^*)$ is an isomorphism with bounded inverse given by the Neumann series

$$(\text{id} + \mathcal{K}_{\pm})^{-1} = \sum_{j=0}^{\infty} (-\mathcal{K}_{\pm})^j.$$

Finally, for each $\varphi \in \mathcal{D}(\Omega, E^*)$, we set $F_{\pm}^{\Omega}(\cdot)[\varphi] := (\text{id} + \mathcal{K}_{\pm})^{-1}(\tilde{\mathcal{R}}_{\pm}(\cdot)[\varphi]) \in \Gamma(E^*)$. Then

$$\begin{aligned} \Gamma(E^*) &\rightarrow E_x^* \\ \varphi &\mapsto F_{\pm}^{\Omega}(x)[\varphi] \end{aligned}$$

is an advanced (retarded) fundamental solution for P at x . Indeed,

$$\begin{aligned} P_{(2)}F_{\pm}^{\Omega}(\cdot)[\varphi] &= F_{\pm}^{\Omega}(\cdot)[P^*\varphi] = (\text{id} + \mathcal{K}_{\pm})^{-1}(\tilde{\mathcal{R}}_{\pm}(\cdot)[P^*\varphi]) \\ &= (\text{id} + \mathcal{K}_{\pm})^{-1}(P_{(2)}\tilde{\mathcal{R}}_{\pm}(\cdot)[\varphi]) = (\text{id} + \mathcal{K}_{\pm})^{-1}(\varphi + \mathcal{K}_{\pm}\varphi) = \varphi \end{aligned}$$

In addition, for each $\varphi \in \mathcal{D}(\Omega, E^*)$, the map $x' \mapsto F_{\pm}^{\Omega}(x')[\varphi]$ is in $\Gamma(\Omega, E^*)$. It can be shown that the approximate fundamental solution $\tilde{\mathcal{R}}_{\pm}(x)$ is in fact an asymptotic expansion of the true fundamental solution $F_{\pm}^{\Omega}(\cdot)(x)$ in a suitable sense. Altogether, we obtain the following result on the solution of the inhomogeneous problem on small domains:

Theorem 2.4. *For any $x \in M$ there exists a relatively compact causal neighborhood Ω such that the following holds: given $v \in \mathcal{D}(\Omega, E)$ and defining u_{\pm} by*

$$\langle u_{\pm}, \varphi \rangle := \int_{\Omega} F_{\pm}^{\Omega}(x)[\varphi] \cdot v(x) dV(x) \quad (\varphi \in \mathcal{D}(\Omega, E))$$

we have: $u_{\pm} \in \Gamma(\Omega, E)$, $Pu_{\pm} = v$, and $\text{supp}(u_{\pm}) \subseteq J_{\pm}^{\Omega}(\text{supp}(v))$.

Turning now to the global theory, the first step is to assure uniqueness of fundamental solutions. For this, we need certain restrictions on the causal structure of M :

Theorem 2.5. *Suppose that M satisfies the causality condition, that the relation \leq is closed on M , and that the time separation function $\tau : M \times M \rightarrow \mathbb{R}$ (see [28]) is finite and continuous. Let $u \in \mathcal{D}'(M, E)$ be a solution of $Pu = 0$ with future or past compact support ($\text{supp}(v) \cap J_{\pm}^M(p)$ compact for all p). Then $u = 0$.*

As an immediate corollary we obtain that under the above assumptions for each $x \in M$ there is at most one fundamental solution for P at x with past (future) compact support. The causality conditions in Th. 2.5 are satisfied if M is globally hyperbolic. Under this assumption, by Th. 2.1 there exists a spacelike Cauchy hypersurface S in M . We denote by $\hat{\xi}$ the future directed timelike unit normal vector field on S and consider the following Cauchy problem:

$$\begin{aligned} Pu &= f && \text{on } M \\ u &= u_0 && \text{on } S \\ \nabla_{\hat{\xi}} u &= u_1 && \text{on } S \end{aligned} \tag{3}$$

where $f \in \mathcal{D}(M, E)$, and $u_0, u_1 \in \mathcal{D}(S, E)$. One first notes that for vanishing f , u_0 , and u_1 , this problem only has the trivial solution. The analytical core of

this result is the observation that for each $\psi \in \Gamma(E^*)$ and each $v \in \Gamma(E)$, one has

$$\psi \cdot (Pv) - (P^*\psi) \cdot v = \operatorname{div}(W) \quad (4)$$

where the vector field W is uniquely characterized by

$$\langle W, X \rangle = (\nabla_X \psi) \cdot v - \psi \cdot (\nabla_X v) \quad (X \in \mathfrak{X}(M)).$$

This allows to control the solution of the homogeneous equation $Pu = 0$ by the Cauchy data on any Cauchy hypersurface. To prove existence of solutions one uses the above local theory to obtain solutions for f , u_0 and u_1 supported in sufficiently small causal domains. More precisely, we call a causal domain Ω an RCCSV-domain (for *relatively compact causal with small volume*) if it is relatively compact and so small that $\operatorname{vol}(\overline{\Omega}) \cdot \|K_{\pm}\|_{\mathcal{C}^0(\overline{\Omega} \times \overline{\Omega})} < 1$. Then we have:

Proposition 2.6. *Let Ω be an RCCSV-domain and suppose that f , u_0 and u_1 are compactly supported in Ω (resp. $\Omega \cap S$). Then the corresponding Cauchy problem in Ω is uniquely solvable.*

Combined with rather subtle causality arguments, this local result finally leads to the following main theorem on existence and uniqueness of solutions to (3):

Theorem 2.7. *Let E be a vector bundle over a globally hyperbolic Lorentz manifold M with a spacelike Cauchy hypersurface S , and let P be normally hyperbolic on $\Gamma(E)$. Then for each $f \in \mathcal{D}(M, E)$ and each $u_0, u_1 \in \mathcal{D}(S, E)$ there exists a unique solution $u \in \Gamma(E)$ satisfying (3). Moreover, $\operatorname{supp}(u) \subseteq J^M(\operatorname{supp}(f) \cup \operatorname{supp}(u_0) \cup \operatorname{supp}(u_1))$ and u depends continuously on (f, u_0, u_1) .*

This result can immediately be utilized to show existence and uniqueness of fundamental solutions:

Theorem 2.8. *Under the assumptions of Th. 2.7, for each $x \in M$ there exists a unique fundamental solution $F_+(x)$ ($F_-(x)$) for P at x with past (future) compact support. These fundamental solutions satisfy*

- (i) $\operatorname{supp}(F_{\pm}(x)) \subseteq J_{\pm}^M(x)$.
- (ii) $\forall \varphi \in \mathcal{D}(M, E^*), x \mapsto F_{\pm}(x)[\varphi] \in \Gamma(E^*)$ and $P^*(F_{\pm}(\cdot)[\varphi]) = \varphi$.

The corresponding fundamental kernels are called Green operators:

Theorem 2.9. *Under the assumptions of Th. 2.7, there exist unique Green operators $G_{\pm} : \mathcal{D}(M, E) \rightarrow \Gamma(M, E)$ satisfying*

- (i) $P \circ G_{\pm} = \operatorname{id}_{\mathcal{D}(M, E)}$
- (ii) $G_{\pm} \circ P|_{\mathcal{D}(M, E)} = \operatorname{id}_{\mathcal{D}(M, E)}$
- (iii) $\forall \varphi \in \mathcal{D}(M, E) : \operatorname{supp}(G_{\pm}\varphi) \subseteq J_{\pm}^M(\operatorname{supp}(\varphi))$

In fact, $(G_{\pm}\varphi)(x) = F_{\mp}(x)[\varphi]$. Moreover, denoting by G_{\pm}^ the Green operators for P^* ,*

$$\int_M (G_{\pm}^*\varphi) \cdot \psi \, dV = \int_M \varphi \cdot (G_{\mp}\psi) \, dV \quad (\varphi \in \mathcal{D}(M, E^*), \psi \in \mathcal{D}(M, E)).$$

3 Wave equations on smooth Lorentzian manifolds: the case of distributional data

In this section we will be concerned with the global Cauchy problem in the case where the metric is still smooth but the data and right hand side are distributional. To keep the presentation simple we will restrict our attention to the wave operator.

Based on Th. 2.1, we will assume that $M = \mathbb{R} \times S$ and the Lorentz metric on M is of the form $\lambda = -\beta dt^2 + g_t$. We denote by \square the d'Alembertian w.r.t. λ . The Cauchy problem we are considering can then be written as

$$\begin{aligned} \square u &= f & \text{on } M \\ u(0, \cdot) &= u_0 \\ \nabla_{\hat{\xi}} u(0, \cdot) &= u_1 \end{aligned} \tag{5}$$

In the present setting, $\hat{\xi} = \frac{1}{\sqrt{\beta}} \partial_t$. To make this initial value problem meaningful in the distributional setting we suppose that f is smooth in the t -variable. More precisely, we assume that $f \in C^\infty(\mathbb{R}, \mathcal{E}'(S)) \cap \mathcal{E}'(\mathbb{R} \times S)$. It then follows from non-characteristic regularity ([23, Th. 8.3.1]) that any $u \in \mathcal{D}'(M)$ with $\square u = f$ has $\pm(1, 0)$ not in the wave front set $\text{WF}(u)|_{(t,x)}$ for all (t, x) . Thus by [13, 23.65.5], $u \in C^\infty(\mathbb{R}, \mathcal{D}'(S))$, so the initial value problem (5) indeed makes sense for $u_0, u_1 \in \mathcal{E}'(S)$.

To our knowledge, (5) has not been treated in full generality in the literature so far for f, u_0, u_1 as specified above. We therefore supply the necessary arguments.

Lemma 3.1. (*Uniqueness*) *There is at most one solution $u \in C^\infty(\mathbb{R}, \mathcal{D}'(S))$ of (5).*

Proof. We first note that \square is strictly hyperbolic with respect to the level sets of the map $T : \mathbb{R} \times S \rightarrow \mathbb{R}, (t, x) \mapsto t$ (in the sense of [22, Def. 23.2.3]). In fact, for the principal symbol σ of \square we have

$$\sigma|_{(t,x)}(\tau, \xi) = \frac{1}{\beta(t, x)} \tau^2 - h_t^{-1}(x)(\xi, \xi).$$

Thus fixing $(t, x) \in M$ and $\xi \in T_x^* S, \xi \neq 0$, the polynomial $p(z) := \sigma|_{(t,x)}(z, \xi)$ has the distinct real zeros $\pm \sqrt{\beta(t, x) h_t^{-1}(\xi, \xi)}$.

Suppose now that u and \tilde{u} are solutions of (5) and set $w := u - \tilde{u}$. By strict hyperbolicity it follows that there exists some neighborhood V of S such that $w|_V = 0$ (see [13, 23.72.8]). Furthermore, by [22, Th. 23.2.9], $\text{WF}(w) \subseteq \text{Char}(\square)$ and is invariant under the Hamiltonian flow of σ . Now $\text{Char}(\square)$ consists entirely of lightlike directions. Thus the projection of any (maximal) bicharacteristic onto M is an inextendible null-geodesic, hence intersects the Cauchy surface S . Since the wavefront set of w is empty in V and is transported along the bicharacteristics it therefore must be empty everywhere on M . Hence $w \in C^\infty(M)$, and $w = 0$ follows from the uniqueness part of Th. 2.7. \square

Turning now to the problem of existence, it clearly suffices to treat the following special cases of (5): on the one hand, the homogeneous problem ($f = 0$), which we denote by (CP1), and on the other hand the inhomogeneous problem with vanishing initial data, called (CP2).

Turning first to (CP2), from Th. 2.9 it is straightforward to conclude that the Green operators $G_{\pm} : \mathcal{D}(M) \rightarrow \mathcal{C}^{\infty}(M)$ continuously extend to operators from $\mathcal{E}'(M) \rightarrow \mathcal{D}'(M)$ as transposed operators of G_{\mp}^* (see [34, Th. 4.3.10]). Thus given $f \in \mathcal{C}^{\infty}(\mathbb{R}, \mathcal{E}'(S)) \cap \mathcal{E}'(\mathbb{R} \times S)$ we may set $w := G_+ f$ to obtain $\square w = f$.

Our task is thereby reduced to proving solvability of (CP1) since adding the solution of (CP1) with $u_0 = -w(0, \cdot)$ and $u_1 = -\nabla_{\xi} w(0, \cdot)$ we obtain the desired solution of (CP2) (note that since $\text{supp}(G_+ f) \subseteq J_+^M(\text{supp } f)$ by [34, (4.3.20)] both u_0 and u_1 are in $\mathcal{E}'(S)$ by [1, Cor. A.5.4]).

To obtain a solution of (CP1) we first observe the following consequence of (4), (cf. [34, Th. 4.3.20]): Denote by $G^* := G_+^* - G_-^*$ the *propagator* of the transposed operator \square^* . Then any smooth solution u of the homogeneous equation $\square u = 0$ satisfies for all $\varphi \in \mathcal{D}(M)$

$$\int_M \varphi \cdot u \, dV = \int_S (\nabla_{\xi} G^*(\varphi)) \cdot u_0 - G^*(\varphi) \cdot u_1 \, dA \quad (6)$$

where $u_0 = u|_S$, $u_1 = \nabla_{\xi} u|_S$, and dA is the Riemannian surface element of the spacelike surface S (i.e., the Riemannian density w.r.t. g_0 in our case). For the distributional Cauchy problem (CP1), we take (6) as a starting point and for given $u_0, u_1 \in \mathcal{E}'(S)$ define $L(u_0, u_1) \in \mathcal{D}'(M)$ by

$$\langle L(u_0, u_1), \varphi \rangle := \langle u_0, (\nabla_{\xi} G^*(\varphi))|_S \rangle - \langle u_1, G^*(\varphi)|_S \rangle.$$

In case u_0 and u_1 are the given Cauchy data $u := L(u_0, u_1)$ will be the desired solution to (CP1). We start establishing this fact by first deriving an explicit formula for $L(u_0, u_1)$ well-suited to the (t, x) -splitting. This naturally has to involve the Green operators. First observe that by Th. 2.9 (ii) we have $\square L(u_0, u_1) = 0$, so the argument preceding Lemma 3.1 shows that in fact $L(u_0, u_1) \in \mathcal{C}^{\infty}(\mathbb{R}, \mathcal{D}'(S))$. Hence for $\varphi_0 \in \mathcal{D}(\mathbb{R})$, $\varphi_1 \in \mathcal{D}(S)$ we may write

$$\langle L(u_0, u_1), \varphi_0 \otimes \varphi_1 \rangle = \int_{\mathbb{R}} \varphi_0(t) \langle L(u_0, u_1)(t), \varphi_1 \rangle \, dt. \quad (7)$$

Denote by $F_{\pm}^*(s, x)$ the fundamental solutions of \square^* at (s, x) according to Th. 2.8. Setting $F_{s,x}^* := F_-^*(s, x) - F_+^*(s, x)$, we have $(G^* \varphi)(s, x) = \langle F_{s,x}^*, \varphi \rangle$ for all $(s, x) \in M$ and all $\varphi \in \mathcal{D}(M)$. Furthermore, $\square F_{s,x}^* = \delta_{(s,x)} - \delta_{(s,x)} = 0$, so $F_{s,x}^* \in \mathcal{C}^{\infty}(\mathbb{R}, \mathcal{D}'(S))$ for each $(s, x) \in M$. Thus for each $t \in \mathbb{R}$ and each $\psi \in \mathcal{D}(S)$ we obtain

$$\langle L(u_0, u_1)(t), \psi \rangle = \langle u_0, \nabla_{\xi} \langle F_{s,x}^*(t), \psi \rangle|_{s=0} \rangle - \langle u_1, \langle F_{0,x}^*(t), \psi \rangle \rangle. \quad (8)$$

We are now in the position to show that u indeed attains the Cauchy data. To this end choose sequences $u_0^{(j)}, u_1^{(j)}$ in $\mathcal{D}(S)$ that converge to u_0 resp. u_1 in $\mathcal{E}'(S)$. By Th. 2.7, for each j there exists a unique $u^{(j)} \in \mathcal{C}^{\infty}(M)$ such that

$$\square u^{(j)} = 0, \quad u^{(j)}(0) = u_0^{(j)}, \quad \nabla_{\xi} u^{(j)}(0) = u_1^{(j)}.$$

In addition, again by [34, Th. 4.3.20] we have $u^{(j)} = L(u_0^{(j)}, u_1^{(j)})$. Moreover, by (8) we obtain for all $\psi \in \mathcal{D}(S)$ and all $t \in \mathbb{R}$

$$\langle u^{(j)}(t), \psi \rangle = \langle L(u_0^{(j)}, u_1^{(j)})(t), \psi \rangle \rightarrow \langle L(u_0, u_1)(t), \psi \rangle = \langle u(t), \psi \rangle, \quad (9)$$

i.e., for all t , $u(t) = \lim u^{(j)}(t)$ in $\mathcal{D}'(S)$. In particular, $u(0) = \lim u^{(j)}(0) = \lim u_0^{(j)} = u_0$, thereby verifying the first initial condition for u .

To show that $\nabla_{\xi} u(0) = u_1$ we first observe that by (8) we have for all $\psi \in \mathcal{D}(S)$ and all $t \in \mathbb{R}$

$$\begin{aligned} \langle \partial_t u^{(j)}(t), \psi \rangle &= \langle L(u_0^{(j)}, u_1^{(j)})'(t), \psi \rangle \\ &\rightarrow \langle L(u_0, u_1)'(t), \psi \rangle = \langle \partial_t u(t), \psi \rangle. \end{aligned}$$

Now since for all $t \in \mathbb{R}$ we have $\nabla_{\xi} u(t) = 1/\sqrt{\beta(t, \cdot)} \partial_t u(t)$ we obtain

$$\begin{aligned} \nabla_{\xi} u(0) &= \frac{1}{\sqrt{\beta(0, \cdot)}} \partial_t u(0) = \frac{1}{\sqrt{\beta(0, \cdot)}} \lim \partial_t u^{(j)}(0) \\ &= \lim \nabla_{\xi} u^{(j)}(0) = \lim u_1^{(j)} = u_1, \end{aligned}$$

thereby also verifying the second initial condition.

Finally, we demonstrate that the support of the unique solution of (5) satisfies the same inclusion relation as in the smooth case (Th. 2.7). To see this, we first note that for any $u \in \mathcal{C}^{\infty}(M)$ such that $\text{supp}(\square u)$ is compact we have the following generalization of (6):

$$\int_M \varphi \cdot u \, dV = \int_M \square u (G_+^* + G_-^*)(\varphi) \, dV + \int_S (\nabla_{\xi} G^*(\varphi)) \cdot u_0 - G^*(\varphi) \cdot u_1 \, dA \quad (10)$$

(this follows by adapting the proof of [1, Lemma 3.2.2]). Now suppose that u is the unique solution of (5) and pick a sequence $u_m \in \mathcal{D}(M)$ such that $u_m(t) \rightarrow u(t)$ for each $t \in \mathbb{R}$. For any $\varphi \in \mathcal{D}(M)$ and each $t \in \mathbb{R}$, $\text{supp}((G_+^* + G_-^*)\varphi(t, \cdot))$ is compact, so

$$\begin{aligned} \langle \square u_m, (G_+^* + G_-^*)\varphi \rangle &= \int_{\mathbb{R}} \langle \square u_m(t), (G_+^* + G_-^*)\varphi(t, \cdot) \rangle \, dt \\ &\rightarrow \int_{\mathbb{R}} \langle \square u(t), (G_+^* + G_-^*)\varphi(t, \cdot) \rangle \, dt = \langle \square u, (G_+^* + G_-^*)\varphi \rangle \end{aligned}$$

Thus applying (10) to each u_m and letting $m \rightarrow \infty$ we obtain

$$\langle u, \varphi \rangle = \langle f, (G_+^* + G_-^*)\varphi \rangle + \langle u_0, (\nabla_{\xi} G^*(\varphi))|_S \rangle - \langle u_1, G^*(\varphi)|_S \rangle.$$

From this and Th. 2.9 (iii), the claimed support properties of u follow. Summing up, we have proved:

Theorem 3.2. *Given $f \in \mathcal{C}^{\infty}(\mathbb{R}, \mathcal{E}'(S)) \cap \mathcal{E}'(\mathbb{R} \times S)$ and $u_0, u_1 \in \mathcal{E}'(S)$ there exists a unique solution $u \in \mathcal{C}^{\infty}(\mathbb{R}, \mathcal{D}'(S))$ of the Cauchy problem (5). Moreover, $\text{supp}(u) \subseteq J^M(\text{supp}(f) \cup \text{supp}(u_0) \cup \text{supp}(u_1))$.*

4 Generalized global analysis

Colombeau algebras of generalized functions ([10, 11]) are differential algebras which contain the vector space of distributions and display maximal consistency with classical analysis in the light of the Schwartz impossibility result ([30]). Here we review global analysis based on the special Colombeau algebra $\mathcal{G}(M)$, for further details see [12, 24, 25] and [19, Sec. 3.2]. The basic idea of its construction is regularization of distributions by nets of smooth functions and the use of asymptotic estimates in terms of a regularization parameter.

Let M be a smooth, second countable Hausdorff manifold. Set $I = (0, 1]$ and denote by $\mathcal{E}(M)$ the subset of $\mathcal{C}^\infty(M)^I$ consisting of all nets depending smoothly on the parameter $\varepsilon \in I$. The algebra of generalized functions on M ([12]) is defined as the quotient $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$ of moderate modulo negligible elements of $\mathcal{E}(M)$, where the respective notions are defined by the following asymptotic estimates. Here \mathcal{P} denotes the space of all linear differential operators on M .

$$\mathcal{E}_M(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(M) : \forall K \subset\subset M \ \forall P \in \mathcal{P} \ \exists N : \sup_{p \in K} |Pu_\varepsilon(p)| = O(\varepsilon^{-N})\}$$

$$\mathcal{N}(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(M) : \forall K \subset\subset M \ \forall P \in \mathcal{P} \ \forall m : \sup_{p \in K} |Pu_\varepsilon(p)| = O(\varepsilon^m)\}.$$

Elements of $\mathcal{G}(M)$ are denoted by $u = [(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(M)$. With componentwise operations, $\mathcal{G}(M)$ is a fine sheaf of differential algebras where the derivations are Lie derivatives with respect to smooth vector fields defined by $L_X u := [(L_X u_\varepsilon)_\varepsilon]$, also denoted by $X(u)$.

There exist embeddings ι of $\mathcal{D}'(M)$ into $\mathcal{G}(M)$ that are sheaf homomorphisms and render $\mathcal{C}^\infty(M)$ a subalgebra of $\mathcal{G}(M)$. Another, more coarse way of relating generalized functions in $\mathcal{G}(M)$ to distributions is based on the notion of association: $u \in \mathcal{G}(M)$ is called associated with $v \in \mathcal{G}(M)$, $u \approx v$, if $u_\varepsilon - v_\varepsilon \rightarrow 0$ in $\mathcal{D}'(M)$. A distribution $w \in \mathcal{D}'(M)$ is called the distributional shadow of u if $u \approx \iota(w)$.

The ring of constants in $\mathcal{G}(M)$ is the space $\tilde{\mathbb{C}}$ of generalized numbers, which form the natural space of point values of Colombeau generalized functions. These, in turn, are uniquely characterized by their values on so-called (compactly supported) generalized points (equivalence classes of bounded nets $(p_\varepsilon)_\varepsilon$ of points, where $(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon$ if $d(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m)$ for each m).

An element $u \in \mathcal{G}(M)$ is called globally bounded, if there exists a representative $(u_\varepsilon)_\varepsilon$ and a $C > 0$ such that $|u_\varepsilon(x)| \leq C$ for all $x \in M$ and all $\varepsilon \in I$.

A similar construction is in fact possible for any locally convex space F : an analogous quotient construction in terms of the seminorms on F allows to assign a $\tilde{\mathbb{C}}$ -module \mathcal{G}_F to F in a natural way ([15]). A particularly important special case is obtained for $F = \Gamma(M, E)$, the space of smooth sections of a vector bundle $E \rightarrow M$ (again with representatives that are supposed to depend smoothly on ε). The resulting space $\Gamma_{\mathcal{G}}(M, E) := \mathcal{G}_F$ then is a $\mathcal{G}(M)$ -module, called the space of generalized sections of the vector bundle E . A convenient

algebraic description is as follows:

$$\Gamma_{\mathcal{G}}(M, E) = \mathcal{G}(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, E) = L_{\mathcal{C}^\infty(M)}(\Gamma(M, E^*), \mathcal{G}(M)). \quad (11)$$

$\Gamma_{\mathcal{G}}$ is a fine sheaf of finitely generated and projective \mathcal{G} -modules. For the special case of generalized tensor fields of rank r, s we use the notation $\mathcal{G}_s^r(M)$. We have:

$$\mathcal{G}_s^r(M) \cong L_{\mathcal{G}(M)}(\mathcal{G}_0^1(M)^s, \mathcal{G}_1^0(M)^r; \mathcal{G}(M)).$$

Observe that this allows the insertion of generalized vector fields and one-forms into generalized tensors, a point of view which is essential when dealing with generalized metrics in the following sense.

Definition 4.1. *An element g of $\mathcal{G}_2^0(M)$ is called a generalized pseudo-Riemannian metric if it is symmetric ($g(\xi, \eta) = g(\eta, \xi) \forall \xi, \eta \in \mathfrak{X}(M)$), its determinant $\det g$ is invertible in \mathcal{G} (equivalently, for each compact subset K of a chart in M there exists some m such that $|\det(g_\varepsilon)_{ij}(p)| > \varepsilon^m$ for ε small and all $p \in K$), and it possesses a well-defined index ν (the index of g_ε equals ν for ε small).*

Based on this definition, many notions from (pseudo-)Riemannian geometry can be extended to the generalized setting (cf. [25]). In particular, any generalized metric induces an isomorphism between generalized vector fields and one-forms, and there is a unique Levi-Civita connection corresponding to g . This provides a convenient framework for non-smooth pseudo-Riemannian geometry and the analysis of highly singular space-times in general relativity (see e.g. [19, Ch. 5], [31]).

For the purposes of the present work we will also need the notion of a time-dependent generalized metric. To this end, let S be an n -dimensional smooth manifold and let $\text{pr}_2: \mathbb{R} \times S \rightarrow S$ denote the projection onto S . Also, let $\text{pr}_2^*(T_2^0 S)$ be the corresponding pullback-bundle.

Definition 4.2. *An element $h \in \Gamma_{\mathcal{G}}(\text{pr}_2^*(T_2^0 S))$ is called t -dependent generalized pseudo-Riemannian metric if h_t possesses a well-defined index and if*

- (i) *(Symmetry) $h_t(\xi, \eta) = h_t(\eta, \xi)$ in $\mathcal{G}(\mathbb{R} \times S)$ for all $\xi, \eta \in \mathfrak{X}(M)$.*
- (ii) *(Non-degeneracy) $\det(h_t)$ is strictly nonzero in the following sense: for any $K \subset \subset \mathbb{R}$, and any L compact in some chart neighborhood on S there exists some m such that $|\det((h_\varepsilon)_t)_{ij}(x)| > \varepsilon^m$ for $(t, x) \in K \times L$ and ε small.*

We conclude this section by the following globalization lemma.

Lemma 4.3. *Let $u: I \times M \rightarrow N$ be a smooth map and let (P) be a property attributable to values $u(\varepsilon, p)$ that is stable with respect to decreasing K and ε in the following sense: if $u(\varepsilon, p)$ satisfies (P) for all $p \in K \subset \subset M$ and all ε less than some $\varepsilon_K > 0$ then for any compact set $K' \subseteq K$ and any $\varepsilon_{K'} \leq \varepsilon_K$, u satisfies (P) on K' for all $\varepsilon \leq \varepsilon_{K'}$. Then there exists a smooth map $\tilde{u}: I \times M \rightarrow N$ such that (P) holds for all $\tilde{u}(\varepsilon, p)$ ($\varepsilon \in I, p \in M$) and for each $K \subset \subset M$ there exists some $\varepsilon_K \in I$ such that $\tilde{u}(\varepsilon, p) = u(\varepsilon, p)$ for all $(\varepsilon, p) \in (0, \varepsilon_K] \times K$.*

Proof. Let $(K_l)_l$ be a compact exhaustion of M with $K_l \subseteq K_{l+1}^\circ$ for all l and choose a smooth function $\eta : M \rightarrow \mathbb{R}$ with $0 < \eta(x) \leq \varepsilon_{K_l}$ for all $x \in K_l \setminus K_{l-1}^\circ$ ($K_0 := \emptyset$) (cf. e.g. [19, Lem. 2.7.3]). Moreover, let $\nu : \mathbb{R}_0^+ \rightarrow [0, 1]$ be a smooth function satisfying $\nu(t) \leq t$ for all t and

$$\nu(t) = \begin{cases} t & 0 \leq t \leq \frac{1}{2} \\ 1 & t \geq \frac{3}{2} \end{cases}$$

For $(\varepsilon, x) \in I \times M$ let $\mu(\varepsilon, x) := \eta(x)\nu\left(\frac{\varepsilon}{\eta(x)}\right)$ and set $\tilde{u}(\varepsilon, x) := u(\mu(\varepsilon, x), x)$. Then \tilde{u} has all the required properties. \square

Remark 4.4. *Lemma 4.3 allows to globalize properties of Colombeau-type generalized functions provided that representatives depend smoothly on ε , as is assumed throughout this work. Indeed, if in the above situation it is additionally assumed that u is a representative of a Colombeau generalized function then by the very nature of the defining asymptotic estimates, \tilde{u} is itself moderate and in fact constitutes a representative of the same generalized function possessing the required property (P) globally on M . In particular, any (time-dependent) generalized metric g possesses a representative $(g_\varepsilon)_\varepsilon$ such that each g_ε is a smooth (time-dependent) metric globally on M , a fact that will repeatedly be used in what is to follow.*

5 Wave equations of non-smooth metrics I: The local theory

In this section we present a local existence and uniqueness result (closely related to the one in [17]) for the homogeneous wave equation of a class of generalized Lorentz metrics g , which will be extended to a global result in Section 6. We start by introducing this class of generalized weakly singular metrics.

5.1 Weakly singular Lorentzian metrics

Let g be a generalized Lorentzian metric on M . From now on we call the pair (M, g) a generalized space-time. To formulate asymptotic conditions on representatives $(g_\varepsilon)_\varepsilon$ of g let m be a background Riemannian metric on M and denote by $\|\cdot\|_m$ the norm induced on the fibers of the respective tensor bundle. To begin with we impose the following condition:

- (A) For all compact sets K , for all orders of derivative $k \in \mathbb{N}_0$ and all k -tuples of smooth vector fields η_1, \dots, η_k and for any representative $(g_\varepsilon)_\varepsilon$ we have:

$$\begin{aligned} \sup_K \|\mathcal{L}_{\eta_1} \dots \mathcal{L}_{\eta_k} g_\varepsilon\|_m &= O(\varepsilon^{-k}) \quad (\varepsilon \rightarrow 0) \\ \sup_K \|\mathcal{L}_{\eta_1} \dots \mathcal{L}_{\eta_k} g_\varepsilon^{-1}\|_m &= O(\varepsilon^{-k}) \quad (\varepsilon \rightarrow 0). \end{aligned} \quad (12)$$

A generalized metric with property (A) will be called a *weakly singular metric*. Note that here we use a somewhat different terminology as compared to [17].

Now with a view to formulating the local Cauchy problem of the wave operator for such a metric we consider a local foliation of M given by the level sets of some non-singular function $t \in C^\infty(U)$, where $U \subseteq M$ is open and relatively compact.

To exclude trivial cases we require the level sets $\Sigma_\tau = \{q \in U : t(q) = \tau\}$ to be space-like with respect to all g_ε . In fact, we suppose a uniform variant of this condition which can also be viewed as a suitable generalization of the classical notion of time-orientability. Moreover, we will need a uniform bound on the covariant derivative of the normal form which, in particular, contains a condition on the second fundamental form of the level sets. More precisely, we demand:

- (B) Each $p \in M$ possesses a neighborhood U on which there exists a *local time-function*, that is a smooth function t with uniformly timelike differential $dt =: \sigma$, i.e.,

$$g_\varepsilon^{-1}(\sigma, \sigma) \leq -C < 0 \quad \text{for some positive constant } C \quad (13)$$

and one (hence any) representative g_ε and all small ε . In addition, we have for all $K \subset\subset U$ that

$$\sup_K \|\nabla^\varepsilon \sigma\|_m = O(1) \quad (\varepsilon \rightarrow 0), \quad (14)$$

where ∇ is the covariant derivative of g .

Let us denote the normal vector field to Σ_τ by ξ (it is a *generalized* vector field defined via its representative $\xi_\varepsilon \in \mathfrak{X}(U)$, given by $\sigma = g_\varepsilon(\xi_\varepsilon, \cdot)$, i.e., $\xi_\varepsilon = \text{grad}_\varepsilon t$). We observe that by pulling up the index, given (A), condition (B) is equivalent to $\sup_K \|\nabla^\varepsilon \xi_\varepsilon\|_m = O(1)$. Also for vector fields X, Y tangent to Σ_τ , we obtain $\nabla^\varepsilon \sigma(X, Y) = Y(\sigma(X)) - \sigma(\nabla_Y^\varepsilon X) = 0 - \sigma(\text{nor} \nabla_Y^\varepsilon X)$, hence we obtain for the second fundamental form II_ε of the hypersurfaces Σ_τ

$$\|II_\varepsilon\|_m = O(1) \quad (\varepsilon \rightarrow 0) \quad \text{uniformly on compact sets.}$$

Remark 5.1. *Conditions (A) and (B) are given in terms of the ε -asymptotics of the generalized metric. There is, however, the following close connection to the classical situation. Assume that we are given a space-time metric that is locally bounded but not necessarily $C^{1,1}$ or of Geroch-Traschen class (i.e., the largest class that allows a consistent distributional treatment, see [16, 26] and [32] for the relation with the present setting). We may then embed this metric into the space of generalized metrics essentially by convolution with a standard mollifier (for details again see [32]). From the explicit form of the embedding it is then clear that condition (A) holds.*

Condition (B), in adapted coordinates, demands somewhat better asymptotics of the time-derivatives of the spatial part of the metric as well as the

spatial derivative of the $(0,0)$ -component. This condition, in fact, is satisfied by several relevant examples as well. In particular, we have:

- Conical space-times fall into our class since estimates (6) and (7) in [33] for the embedded metric imply our condition (A), while (B) is immediate from the staticity of the metric.
- Impulsive pp-waves (in “Rosen form”)

$$ds^2 = -dudv + (1 + u_+)^2 dx^2 + (1 - u_+)^2 dy^2$$

as well as expanding impulsive waves with line element

$$ds^2 = 2dudv + 2v^2 \left| dz + \frac{u_+}{2v} \overline{H} d\bar{z} \right|^2$$

satisfy conditions (A) and (B). Here u_+ denotes the kink function and $H(z)$ is the Schwarzian derivative $H = h'''/(2h') - (3h''^2)/(4h'^2)$ of some analytic function $h(z)$ (which may be chosen arbitrarily). For details see [18, Ch. 20]. In both cases the metric is continuous and it will obey conditions (A) and (B) when embedded with a standard mollifier, or – more generally – if we use any regularization that converges locally uniformly to the original metric.

5.2 Local existence and uniqueness

We start by formulating the local Cauchy problem for the wave operator on weakly singular space-times. Let $p \in M$, and choose U to be an open and relatively compact neighborhood of p as in condition (B). Denote the corresponding foliation by $\Sigma_\tau = \{q \in U : t(q) = \tau\}$ ($\tau \in [-\gamma, \gamma]$) and suppose $p \in \Sigma_0 =: \Sigma$. In addition to the normal vector field ξ and the normal covector field σ whose asymptotics have already been discussed above we will need their corresponding normalized versions $\hat{\xi} = [(\hat{\xi}_\varepsilon)_\varepsilon] = [(\xi_\varepsilon/V_\varepsilon)_\varepsilon]$ and $\hat{\sigma} = [(\hat{\sigma}_\varepsilon)_\varepsilon] = g(\hat{\xi}, \cdot)$, where we have set $V_\varepsilon^2 = -g_\varepsilon(\xi_\varepsilon, \xi_\varepsilon)$.

We are interested in the initial value problem

$$\begin{aligned} \square u &= 0 & \text{on } U \\ u &= u_0 & \text{on } \Sigma \\ \nabla_{\hat{\xi}} u &= u_1 & \text{on } \Sigma, \end{aligned} \tag{15}$$

where the initial data u_0, u_1 are supposed to be in $\mathcal{G}(\Sigma)$. Note that this, in particular, includes the case of distributional initial data. We are interested in finding a local solution $u \in \mathcal{G}$ on U or an open subset thereof.

A general strategy to solve PDEs in generalized functions is the following. First, solve the equation for fixed ε in the smooth setting and form the net $(u_\varepsilon)_\varepsilon$ of smooth solutions. This will be a candidate for a solution in \mathcal{G} , but particular care has to be taken to guarantee that the u_ε share a common domain

of definition and depend smoothly on ε . In fact, as has recently been shown in [5] it suffices to verify continuous dependence on ε . In the second step, one shows that the solution candidate $(u_\varepsilon)_\varepsilon$ is a moderate net, hence obtaining existence of a solution $[(u_\varepsilon)_\varepsilon]$ in \mathcal{G} . Finally, to obtain uniqueness of solutions, one has to prove that changing representatives of the data leads to a solution that is still in the class $[(u_\varepsilon)_\varepsilon]$. Note that this amounts to an additional stability of the equation with respect to negligible perturbations of the initial data.

So, in the present situation we need a condition which provides us with the existence of a solution candidate:

- (C) For each $p \in \Sigma$ there exists a neighborhood $V \subseteq U$ and a representative $(g_\varepsilon)_\varepsilon$ of the metric g on V such that V is, for each ε , an RCCSV-neighborhood in (M, g_ε) with $\Sigma \cap V$ a spacelike Cauchy hypersurface for V .

Indeed Prop. 2.6 now provides us with a solution candidate defined on V , that is a net (u_ε) with $\square_\varepsilon u_\varepsilon = f_\varepsilon$ on V for some negligible net (f_ε) and moreover $u_\varepsilon|_{\Sigma \cap V} = u_{0\varepsilon}$, $\nabla_\xi u|_{\Sigma \cap V} = u_{1\varepsilon}$ for some representatives $(u_{0\varepsilon})$, $(u_{1\varepsilon})$ of the data. We note that continuous dependence of u on ε follows readily from the construction steps detailed in Section 2. Observe that the only part that exceeds the classical condition for existence and uniqueness of solutions is a certain uniformity in ε . Heuristically this means that the light-cones of the metric g_ε neither vary to wildly with ε nor collapse as $\varepsilon \rightarrow 0$. In terms of regularizations of classical metrics which are locally bounded but not necessarily of $\mathcal{C}^{1,1}$ or of Geroch-Traschen class, this condition will always be satisfied due to non-degeneracy of the classical metric.

Now we may state the main result of this section.

Theorem 5.2 (Local existence and uniqueness of generalized solutions). *Let (M, g) be a generalized space-time with a weakly singular metric and assume that conditions (B) and (C) hold. Then, for each $p \in \Sigma$ there exists an open neighborhood Ω such that for all compactly supported $u_0, u_1 \in \mathcal{G}(\Sigma \cap \Omega)$, the initial value problem (15) has a unique solution u in $\mathcal{G}(\Omega)$.*

The core of the proof of Th. 5.2 consists of higher order energy estimates for the solution candidate whose existence is secured by condition (C). The energy estimates, which rely on conditions (A) and (B), will be carried out in a geometric setting using ε -dependent energy momentum tensors and ε -dependent Sobolev norms. These notions have been introduced in [33] and a suitable generalization of this method will be presented in the next section.

5.3 Higher order energy estimates

Let $U \supseteq V$ be the neighborhoods of p given by conditions (B) and (C). The solution candidate $(u_\varepsilon)_\varepsilon$ is defined on V and we are going to estimate u_ε on some suitable neighborhood of p .

We start by introducing some notation. Let Ω be an open neighborhood of p with the property that $\overline{\Omega} \subseteq V$, and such that the boundary of the region

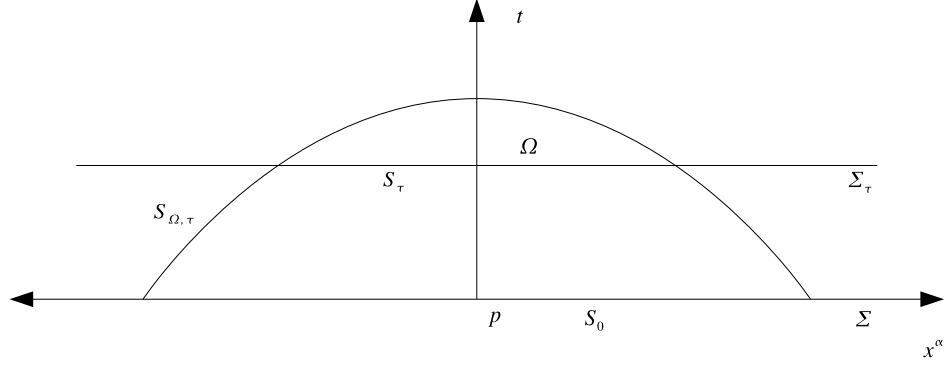


Figure 1: Local foliation of space-time

$\Omega \cap \{q \in U : t(q) \geq 0\}$ and the boundary of the region $\Omega \cap \{q \in U : t(q) \leq 0\}$ is space-like w.r.t. all g_ε . Indeed, such a neighborhood exists by condition (A). We now concentrate on the forward-in-time part of Ω (i.e., the part where $t \geq 0$) since the backward-in-time part can be dealt with analogously. To this end we set $S_\tau := \Sigma_\tau \cap \Omega$ and denote by Ω_τ the open part of Ω between Σ and Σ_τ . We denote the part of the boundary of Ω_τ with $0 < t < \tau$ by $S_{\Omega, \tau}$, so that $\partial\Omega_\tau = S_0 \cup S_\tau \cup S_{\Omega, \tau}$ (see Figure 1).

From now on we will adopt abstract index notation for (generalized) tensorial objects (see e.g. [29]). In particular, representatives of the metric g_ε and its inverse will be denoted by g_{ab}^ε and g_ε^{ab} , respectively. In addition, to simplify the notation for tensors we are going to use capital letters to abbreviate tuples of indices, i.e., we will write T_J^I for $T_{q_1 \dots q_s}^{p_1 \dots p_r}$ with $|I| = r$, $|J| = s$.

We now define a generalized Riemannian metric by

$$e := g + 2\hat{\sigma} \otimes \hat{\sigma},$$

and use it in combination with the covariant derivative ∇ of g to define “ ε -dependent” Sobolev norms and energies on U . Observe that by conditions (A) and (B) we have $\|e_\varepsilon\|_m = O(1)$ and $\|\nabla_\varepsilon e_\varepsilon\|_m = O(1)$ on compact sets. We will also frequently need tensor products of e and use the notation $e_{IJ} = e_{p_1 q_1} \dots e_{p_r q_r}$ with $|I| = r = |J|$.

Definition 5.3 (Sobolev norms and energies). *Let T_J^I be a smooth tensor field and u a smooth function on U , $0 \leq \tau \leq \gamma$, and $k, j \in \mathbb{N}_0$.*

- (i) *We define the pointwise norm of T_J^I by $\|T_J^I\|_{e_\varepsilon}^2 := e_{KL}^\varepsilon e_\varepsilon^{IJ} T_I^K T_J^L$ and the higher order pointwise norm of u by $|\nabla_\varepsilon^{(j)} u|^2 := \|\nabla_{p_1}^\varepsilon \dots \nabla_{p_j}^\varepsilon u\|_{e_\varepsilon}^2$.*

(ii) On Ω_τ and S_τ we define the Sobolev norms

$$\begin{aligned} \nabla \|u\|_{\Omega_\tau, \varepsilon}^k &:= \left(\sum_{j=0}^k \int_{\Omega_\tau} |\nabla_\varepsilon^{(j)}(u)|^2 \mu_\varepsilon \right)^{\frac{1}{2}}, \text{ and} \\ \nabla \|u\|_{S_\tau, \varepsilon}^k &:= \left(\sum_{j=0}^k \int_{S_\tau} |\nabla_\varepsilon^{(j)}(u)|^2 \mu_\tau^\varepsilon \right)^{\frac{1}{2}} \end{aligned}$$

respectively. Here $\mu = [(\mu^\varepsilon)]$ and $\mu_\tau = [(\mu_\tau^\varepsilon)]$ denote the respective volume forms on Ω_τ and S_τ derived from g . Note that although in the second norm the integration is performed over the three-dimensional manifold S_τ only, derivatives are not confined to directions tangential to S_τ .

(iii) On Ω we define the energy momentum tensors by ($k > 0$)

$$\begin{aligned} T_\varepsilon^{ab,0}(u) &:= -\frac{1}{2} g_\varepsilon^{ab} u^2, \\ T_\varepsilon^{ab,k}(u) &:= \left(g_\varepsilon^{ac} g_\varepsilon^{bd} - \frac{1}{2} g_\varepsilon^{ab} g_\varepsilon^{cd} \right) e_\varepsilon^{p_1 q_1} \dots e_\varepsilon^{p_{k-1} q_{k-1}} \\ &\quad \times (\nabla_c^\varepsilon \nabla_{p_1}^\varepsilon \dots \nabla_{p_{k-1}}^\varepsilon u) (\nabla_d^\varepsilon \nabla_{q_1}^\varepsilon \dots \nabla_{q_{k-1}}^\varepsilon u), \end{aligned}$$

(iv) Finally, the energy integrals are defined by

$$E_{\tau, \varepsilon}^k(u) := \sum_{j=0}^k \int_{S_\tau} T_\varepsilon^{ab,j}(u) \xi_a \xi_b V_\varepsilon^{-1} \mu_\tau^\varepsilon. \quad (16)$$

A straightforward calculation shows that the tensor fields $T_\varepsilon^{ab,k}(u)$ satisfy the dominant energy condition hence an application of Stokes' theorem yields the basic energy estimate (see e.g. [21, Sec. 4.3])

$$E_{\tau, \varepsilon}^k(u) \leq E_{\tau=0, \varepsilon}^k(u) + \sum_{j=0}^k \int_{\Omega_\tau} (\xi_b \nabla_a^\varepsilon T_\varepsilon^{ab,j}(u) + T_\varepsilon^{ab,j}(u) \nabla_a^\varepsilon \xi_b) \mu_\varepsilon. \quad (17)$$

One key estimate in our approach is the equivalence of Sobolev norms and energies. Indeed using condition (A) and the estimate (13) one may derive ([17, Lemma 4.1]):

Lemma 5.4 (Energy integrals and Sobolev norms). *There exist constants C, C' such that for each $k \geq 0$ and all ε small*

$$C(\nabla \|u\|_{S_\tau, \varepsilon}^k)^2 \leq E_{\tau, \varepsilon}^k(u) \leq C'(\nabla \|u\|_{S_\tau, \varepsilon}^k)^2. \quad (18)$$

With this tool at hand we may derive the core estimate allowing to prove existence and uniqueness of solutions.

Proposition 5.5. *Let (u_ε) be a solution candidate on V . Then, for each $k \geq 1$, there exist positive constants C'_k, C''_k, C'''_k such that for all $0 \leq \tau \leq \gamma$ we have*

$$\begin{aligned} E_{\tau,\varepsilon}^k(u_\varepsilon) &\leq E_{0,\varepsilon}^k(u_\varepsilon) + C'_k(\nabla \|f_\varepsilon\|_{\Omega_{\tau,\varepsilon}}^{k-1})^2 + C''_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_0^\tau E_{\zeta,\varepsilon}^j(u_\varepsilon) d\zeta \\ &\quad + C'''_k \int_0^\tau E_{\zeta,\varepsilon}^k(u_\varepsilon) d\zeta. \end{aligned} \quad (19)$$

Before sketching the proof of this statement, we draw the essential conclusions from it. Observe that the constant in front of the highest order term on the r.h.s. does not depend on ε , hence we obtain, by an application of Gronwall's lemma:

Corollary 5.6. *Let (u_ε) be a solution candidate on V . Then, for each $k \geq 1$, there exist positive constants C'_k, C''_k, C'''_k such that for all $0 \leq \tau \leq \gamma$,*

$$\begin{aligned} E_{\tau,\varepsilon}^k(u_\varepsilon) &\leq \left(E_{0,\varepsilon}^k(u_\varepsilon) + C'_k(\nabla \|f_\varepsilon\|_{\Omega_{\tau,\varepsilon}}^{k-1})^2 + C''_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_{\zeta=0}^\tau E_{\zeta,\varepsilon}^j(u_\varepsilon) d\zeta \right) e^{C'''_k \tau} \end{aligned} \quad (20)$$

Consequently, if the initial energy $(E_{0,\varepsilon}^k(u_\varepsilon))_\varepsilon$ is a moderate (resp. negligible) net of real numbers, and $(f_\varepsilon)_\varepsilon$ is negligible then

$$\sup_{0 \leq \tau \leq \gamma} (E_{\tau,\varepsilon}^k(u_\varepsilon))_\varepsilon$$

is moderate (resp. negligible).

Now we sketch the proof of Proposition 5.5: We have to estimate the right hand side of the basic energy estimate (17). Starting with the second term under the integral we use condition (B) to obtain

$$|T_\varepsilon^{ab,j}(u_\varepsilon) \nabla_a^\varepsilon \xi_b| \leq \|T_\varepsilon^{ab,j}(u_\varepsilon)\|_{e_\varepsilon} \|\nabla_a^\varepsilon \xi_b\|_{e_\varepsilon} \leq C \|T_\varepsilon^{ab,j}(u_\varepsilon)\|_{e_\varepsilon}. \quad (21)$$

Observe that by condition (A), the Riemannian metric e is $O(1)$ hence $\|\cdot\|_m$ and $\|\cdot\|_{e_\varepsilon}$ are equivalent norms. We now may estimate $\|T_\varepsilon^{ab,j}(u_\varepsilon)\|_{e_\varepsilon}$ by the higher order pointwise norm of u . After integration this gives

$$\left| \sum_{j=0}^k \int_{\Omega_\tau} T_\varepsilon^{ab,j}(u_\varepsilon) \nabla_a^\varepsilon \xi_b \mu_\varepsilon \right| \leq C (\nabla \|u_\varepsilon\|_{\Omega_{\tau,\varepsilon}}^k)^2. \quad (22)$$

Now turning to the divergence term on the right hand side of (17) we start with orders $k = 0, 1$. Using the wave equation we find

$$\nabla_a^\varepsilon T_\varepsilon^{ab,0}(u_\varepsilon) = -u_\varepsilon \nabla_\varepsilon^b u_\varepsilon \quad \text{and} \quad \nabla_a^\varepsilon T_\varepsilon^{ab,1}(u_\varepsilon) = f_\varepsilon \nabla_\varepsilon^b u_\varepsilon,$$

which after integration clearly can be estimated by the squares of $\nabla \|u_\varepsilon\|_{\Omega_\tau, \varepsilon}^1$ and $\nabla \|f_\varepsilon\|_{\Omega_\tau, \varepsilon}^0$. Inserting this and (22) for $k = 1$ into (17) we obtain

$$E_{\tau, \varepsilon}^1(u_\varepsilon) \leq E_{0, \varepsilon}^1(u_\varepsilon) + C \left(\nabla \|f_\varepsilon\|_{\Omega_\tau, \varepsilon}^0 \right)^2 + C \left(\nabla \|u_\varepsilon\|_{\Omega_\tau, \varepsilon}^1 \right)^2.$$

Next we use Lemma 5.4 to estimate the Sobolev norm of u_ε in terms of its energy, i.e.,

$$\left(\nabla \|u_\varepsilon\|_{\Omega_\tau, \varepsilon}^1 \right)^2 = \int_0^\tau \left(\nabla \|u_\varepsilon\|_{S_\zeta, \varepsilon}^1 \right)^2 d\zeta \leq C \int_0^\tau E_{\zeta, \varepsilon}^1(u_\varepsilon) d\zeta,$$

which gives the claim for $k = 1$ (with $C_1'' = 0$).

Finally, one has to estimate the divergence of the higher (i.e., $k > 1$) order energy momentum tensors. The general strategy is, of course, to rewrite terms containing the $(k + 1)^{\text{st}}$ order derivative of u_ε using the wave equation. This necessitates interchanging the order of covariant derivatives, which introduces additional curvature terms. These can be estimated using condition (A). Observe, however, that there also appear terms where the covariant derivatives falls on e . These terms of the form $\nabla_a^\varepsilon e_\varepsilon^{IJ}$ can be estimated thanks to condition (B). \square

We finally *sketch the proof of Th. 5.2*: We have already noted the existence of a solution candidate (u_ε) on V . To prove that u_ε is moderate on Ω_γ we start from moderateness of the data u_0, u_1 . Inductively using the wave equation this translates into moderateness of the initial energies $(E_{0, \varepsilon}^k(u_\varepsilon))_\varepsilon$. Now by Corollary 5.6 we obtain moderateness of the energies $(E_{\tau, \varepsilon}^k(u_\varepsilon))_\varepsilon$ for all $0 \leq \tau \leq \gamma$. Finally, we use the Sobolev embedding theorem (together with the fact the volume is $O(1)$ due to condition (A)) to estimate the sup-norm of u_ε in terms of the Sobolev norms, which in turn can be bounded by the energies $(E_{\tau, \varepsilon}^k(u_\varepsilon))_\varepsilon$ due to Lemma 5.4 (for details see [17, Lemma 6.2]). So we see that moderateness of the energies implies moderateness of (u_ε) and we have proved existence of solutions.

Uniqueness follows along the same lines replacing moderateness by negligibility. \square

6 Wave equations of non-smooth metrics II: The global theory

We extend the results of Th. 5.2 to establish existence and uniqueness of global generalized solutions. As in the classical situation we have to impose additional global conditions on the generalized Lorentzian metric to control causality properties of space-time in the large. Thus, we begin by transferring the notion of global hyperbolicity to the setting of generalized space-times by appealing to a variant in terms of the metric splitting property stated in Th. 2.1, (iii).

Definition 6.1. Let g be a generalized Lorentz metric on the smooth $(n + 1)$ -dimensional manifold M . We say that (M, g) allows a globally hyperbolic metric splitting if there exists a C^∞ -diffeomorphism $\psi: M \rightarrow \mathbb{R} \times S$, where S is an n -dimensional smooth manifold such that the following holds for the pushed forward generalized Lorentz metric $\lambda := \psi_*g$ on $\mathbb{R} \times S$:

(a) There is a representative $(\lambda_\varepsilon)_{\varepsilon \in I}$ of λ such that every λ_ε is a Lorentz metric and each slice $\{t_0\} \times S$ with arbitrary $t_0 \in \mathbb{R}$ is a (smooth, spacelike) Cauchy hypersurface for every λ_ε ($\varepsilon \in I$).

(b) We have the metric splitting of λ in the form

$$\lambda = -\beta dt^2 + h,$$

where $h \in \Gamma_{\mathcal{G}}(\text{pr}_2^*(T_2^0 S))$ is a t -dependent generalized Riemannian metric (in the sense of Def. 4.2) and $\beta \in \mathcal{G}(\mathbb{R} \times S)$ is globally bounded and locally uniformly positive, i.e., for some (hence any) representative (β_ε) of β and for every $K \subset \subset \mathbb{R} \times S$ we can find a constant $C > 0$ such that $\beta_\varepsilon(x) \geq C$ holds for small $\varepsilon > 0$ and $x \in K$.

(c) For every $T > 0$ there exists a representative (h_ε) of h and a smooth complete Riemannian metric ρ on S which uniformly bounds h from below in the following sense: for all $t \in [-T, T]$, $x \in S$, $v \in T_x S$, and $\varepsilon \in I$

$$(h_\varepsilon)_t(v, v) \geq \rho(v, v).$$

Remark 6.2. Observe that the basic splitting structure and the requirements on lower bounds for β and h in the above definition display common features with the notion of regularly sliced space-times ([6, Ch. XII, Subsec. 11.4]) which provide sufficient conditions for global hyperbolicity in the smooth case ([7]).

Example 1. To obtain simple non-trivial examples of generalized space-times satisfying the conditions of Def. 6.1 we consider Robertson-Walker space-times. First, we briefly recall the classical situation: Let (S, h_0) be a connected Riemannian manifold, $f: \mathbb{R} \rightarrow]0, \infty[$ be smooth, and put $\lambda_{(t,x)} = -dt^2 + f(t)^2(h_0)_x$ for every $(t, x) \in \mathbb{R} \times S$. Then the Lorentzian metric λ on $\mathbb{R} \times S$ is globally hyperbolic if and only if (S, h_0) is complete (cf. [1, Lemma A.5.14]). Moreover, if this is the case, then every slice $\{t_0\} \times S$ is a smooth, spacelike Cauchy hypersurface.

We generalize the Robertson-Walker space-time by allowing as warping function any $f \in \mathcal{G}(\mathbb{R})$ that is globally bounded and locally uniformly positive and replace h_0 by a generalized Riemannian metric on S which is bounded below by some smooth complete Riemannian metric (to guarantee condition (c)). By Lemma 4.3 on globalization techniques we may pick representatives (f_ε) of f and $(h_{0\varepsilon})$ of h_0 such that the smooth function f_ε is everywhere positive and $h_{0\varepsilon}$ is a Riemannian metric on S for every $\varepsilon \in I$. In addition, we assume that each $h_{0\varepsilon}$ is complete. The generalized Lorentz metric $\lambda := -dt^2 + f^2 h_0$ on $\mathbb{R} \times S$ then trivially satisfies condition (b). Putting $\lambda_\varepsilon := -dt^2 + f_\varepsilon^2 h_{0\varepsilon}$ ($\varepsilon \in I$) we obtain a representative of λ . By completeness of the Riemann metric $h_{0\varepsilon}$, the

Lorentz metric λ_ε is globally hyperbolic and every slice $\{t_0\} \times S$ is a Cauchy hypersurface for every $\varepsilon \in I$. Thus condition (a) in Def. 6.1 is also satisfied.

From now on we consider only generalized space-times (M, g) which possess a globally hyperbolic metric splitting. To simplify notation we will henceforth suppress the diffeomorphism providing the splitting and assume that $M = \mathbb{R} \times S$ and $g = \lambda$ with S and λ as in the statement of Def. 6.1. Thus, the generalized space-time is represented by a family of globally hyperbolic space-times (M, g_ε) such that $S \cong \{0\} \times S$ is a Cauchy hypersurface for every g_ε ($\varepsilon \in I$).

Therefore we are provided with a suitable Cauchy hypersurface for the initial value problem for the wave operator \square corresponding to the generalized space-time metric g on M , i.e., the Cauchy problem

$$\begin{aligned} \square u &= 0 & \text{on } M \\ u &= u_0 & \text{on } S \\ \nabla_{\hat{\xi}} u &= u_1 & \text{on } S. \end{aligned} \tag{23}$$

Here the unit normal vector field of S is given by $\hat{\xi} = \frac{1}{\sqrt{\beta}} \partial_t$ and the initial data u_0, u_1 are assumed to belong to $\mathcal{G}(S)$ and to have compact supports, e.g., arising by embedding distributional data from $\mathcal{E}'(S)$.

Now the key strategy to establish a global version of Th. 5.2 on existence and uniqueness of solutions to the Cauchy problem (23) is as follows: From the classical existence and uniqueness result in Th. 2.7 for every $\varepsilon \in I$ we obtain a solution candidate defined on all of M , which again depends continuously on ε . In this sense the global hyperbolic metric splitting of (M, g) replaces condition (C) used in the proof of Th. 5.2 to produce a solution candidate. Then we aim at showing moderateness, thus existence of a generalized solution, as well as uniqueness by employing energy estimates as in the local constructions of Section 5. Therefore it is appropriate to suppose also condition (A), i.e., g to be weakly singular. In the present situation we may translate (12) into corresponding asymptotic conditions on β and h_t . As for condition (B), we see that the existence of a suitable (in this case even global) foliation is a consequence of the globally hyperbolic metric splitting. Indeed we globally have $g_\varepsilon^{-1}(dt, dt) = -1/\beta_\varepsilon \leq -C < 0$ for some positive C , which implies (13). The second asymptotic boundedness condition (14) in (B), i.e., $\sup_K \|\nabla^\varepsilon dt\|_m = O(1)$, now simply reads

$$\|d\beta_\varepsilon\|_m = O(1) \quad (\varepsilon \rightarrow 0) \tag{24}$$

uniformly on compact sets. As in the local setting this implies for the extrinsic curvature of the hypersurfaces $\{t\} \times S$

$$\|II_\varepsilon\|_m = O(1) \quad (\varepsilon \rightarrow 0) \quad \text{uniformly on compact sets.}$$

We may now state the main result of this section.

Theorem 6.3 (Global existence and uniqueness of generalized solutions). *Let (M, g) be a generalized space-time with a weakly singular metric admitting a*

globally hyperbolic metric splitting and assume that condition (14) holds. Then the Cauchy problem (23) has a unique solution $u \in \mathcal{G}(M)$ for all compactly supported $u_0, u_1 \in \mathcal{G}(S)$.

Sketch of proof: Let \square_ε denote the wave operator corresponding to g_ε . Then Th. 2.7 provides us with a global solution u_ε to $\square_\varepsilon = 0$ with Cauchy data $u_{0\varepsilon}$ and $u_{1\varepsilon}$, thereby defining a solution candidate.

To prove existence we have to establish moderateness of the net $(u_\varepsilon)_\varepsilon$. Choose an exhaustive sequence of compact sets K_j ($j \in \mathbb{N}$) in S . Then it suffices to show moderateness of $(u_\varepsilon)_\varepsilon$ on $L_j := [-j, j] \times K_j$ for each $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$ and choose ρ as in Def. 6.1, (c) with $T = j$. We cover L_j by finitely many lens-shaped regions as in Figure 1 that are, in turn, contained in coordinate neighborhoods. By the explicit construction given in [27, Sec. 3.3.4] (based on condition (A)) the heights of these lenses are uniformly bounded below on L_j by some $\delta > 0$ due to the properties of ρ and β .

Employing the energy estimates from Section 4.3 we thereby derive moderateness estimates of u_ε on the strip $[0, \delta/2] \times K_j$ from the moderateness of $u_{0\varepsilon}$ and $u_{1\varepsilon}$. We may iterate this procedure to cover $[0, j] \times K_j$ in finitely many steps, and analogously for $[-j, 0] \times K_j$.

By the same token negligibility of $(u_{0\varepsilon})$ and $(u_{1\varepsilon})$ implies the corresponding property for (u_ε) , which proves uniqueness. \square

Finally, we remark that our methods also allow to treat the inhomogeneous equation as well as the inclusion of lower order terms, which have to satisfy certain asymptotic bounds, see [20].

Also, condition (A) is actually a little stronger than what is needed to prove moderateness of the solution candidate resp. negligibility in case of negligible data. Indeed, again by a result of [20], it suffices to suppose (A) for $k = 0$. On the other hand we could use (A) to explicitly calculate ε -power bounds of (derivatives) of the solution, which encode additional regularity information of our generalized solutions.

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